



IIT ROORKEE



NPTEL ONLINE
CERTIFICATION COURSE

QUANTITATIVE INVESTMENT MANAGEMENT

LECTURE 20

Duration Properties

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DURATION OF ZEROS

- The duration of a zero coupon bond is equal to its maturity.
- Hence, longer term zeros are more price sensitive than short term zeros.
- This makes intuitive sense, since a change in yield for a long term bond affects cash flows over a larger number of discounting periods.



DURATION OF A PERPETUITY

$$D_{\text{perpetuity}} = \frac{\sum_{t=1}^{\infty} tC_t (1+y)^{-t}}{\sum_{t=1}^{\infty} C_t (1+y)^{-t}} = \frac{\sum_{t=1}^{\infty} ct (1+y)^{-t}}{\sum_{t=1}^{\infty} c(1+y)^{-t}}$$

$$= \frac{\sum_{t=1}^{\infty} t(1+y)^{-t}}{\sum_{t=1}^{\infty} (1+y)^{-t}} = \frac{\sum_{t=1}^{\infty} t(1+y)^{-t}}{1/y}$$

$$\text{Now let } S = \sum_{t=1}^{\infty} t(1+y)^{-t}. \text{ Then } S(1+y)^{-1} = \sum_{t=1}^{\infty} t(1+y)^{-(t+1)}$$

$$\begin{aligned}
\text{Hence, } S \left[1 - (1 + y)^{-1} \right] &= \sum_{t=1}^{\infty} t(1 + y)^{-t} - \sum_{t=1}^{\infty} t(1 + y)^{-(t+1)} \\
&= \sum_{t=0}^{\infty} (t + 1)(1 + y)^{-(t+1)} - \sum_{t=1}^{\infty} t(1 + y)^{-(t+1)} \\
&= \sum_{t=0}^{\infty} t(1 + y)^{-(t+1)} + \sum_{t=0}^{\infty} (1 + y)^{-(t+1)} - \sum_{t=1}^{\infty} t(1 + y)^{-(t+1)} \\
&= \sum_{t=0}^{\infty} (1 + y)^{-(t+1)} = \sum_{t=1}^{\infty} (1 + y)^{-t} = \frac{1}{y}. \text{ Hence, } S = \frac{1 + y}{y^2} \text{ and} \\
D &= \frac{(1 + y)/y^2}{1/y} = \frac{1 + y}{y}
\end{aligned}$$

DURATION:IMPORTANT PROPERTIES

- The Macaulay Duration of a zero equals its maturity.
- Macaulay Duration of a perpetuity does not depend on coupon rates ($D=(1+y)/y$)..



DURATION AND COUPON RATES

- For practically realizable values, duration of all types of bonds (par, premium and discount) **decreases with increase in coupon rate.**
- This is because as coupon increases a greater proportion of the cashflows are realized by the investor earlier.



As an illustration, we consider the case of a 5 year Rs 1,000/- bond with a YTM of 20%. The values of the various measures of interest rate sensitivity are tabulated below:

Coupon Rate (%)	DV01(Rs/%)	D _{Mod} (Years)	D _{Mac} (Years)	Price (Rs)
5	20.03	3.63	4.36	551
10	23.32	3.33	3.99	701
15	26.61	3.13	3.76	850
20	29.90	2.99	3.59	1000
25	33.20	2.89	3.47	1150
30	36.48	2.81	3.37	1299



DURATION AND MATURITY

- Duration of **par and premium bonds always increases with increasing maturity.**
- For discount bonds, there is a critical value of maturity, upto which duration would increase with maturity.
- If maturity exceeds this critical value, then duration will start decreasing with maturity for discount bonds.



As an illustration of this phenomenon, we consider a Rs 1,000 face value bond with a coupon rate of 15% quoting at a YTM of 25%. The above data corresponds to a $T_c = 11.5$ years.

Maturity (Years)	DV01(Rs/%)	D _{Mod} (Years)	D _{Mac} (Years)	D*
3	16.63	2.06	2.58	-6.8
5	21.38	2.92	3.66	-5.2
10	24.86	3.87	4.83	-1.2
20	24.46	4.05	5.06	6.8
25	24.21	4.03	5.03	10.8
50	24.00	4.00	5.00	30.8

DURATION WITH CONTINUOUS COMPOUNDING

$$P(y) = \sum_t C_t e^{-yt}$$

$$P'(y) = - \sum_t t C_t e^{-yt}$$

$$\begin{aligned} \frac{dP}{P} &= \frac{P'(y)}{P(y)} dy = - \frac{\sum_t t C_t e^{-yt}}{P} dy \\ &= -D dy \quad \text{where } D = \frac{\sum_t t C_t e^{-yt}}{P} \end{aligned}$$

PRICE SENSITIVITY

Price Sensitivity of a Bond

$$DV01 = -\frac{dP}{dy} = D_{MOD} \times P = \frac{D_{MAC}}{1+y} \times P$$

- Hence, two effects interact due to interest rate shifts:
- DURATION EFFECT
- PRICE EFFECT



PRICE SENSITIVITY VS MATURITY OF PREMIUM BONDS

- The duration effect of par & premium bonds increases with maturity.
- The price of a premium bond increases with maturity while that of a par bond remains constant.
- It follows that price sensitivity of premium and par bonds will always increase with maturity.



PRICE SENSITIVITY & MATURITY OF DISCOUNT BONDS

- For discount bonds, there is a critical value of maturity, upto which duration would increase with maturity.
- If maturity exceeds this critical value, then duration will start decreasing with maturity for discount bonds.
- The price of a discount bond decreases with maturity.

- For short term discount bonds, the duration effect dominates and sensitivity increases to start with.
- As maturity increases, the decrease due to price effect also becomes significant and the sensitivity starts decreasing gradually until a limiting value corresponding to a perpetuity is reached.

PRICE VOLATILITY

$$\text{Price Volatility of a Bond} = -\frac{dP/P}{dy} = D_{\text{MOD}} = \frac{D_{\text{MAC}}}{1+y}$$

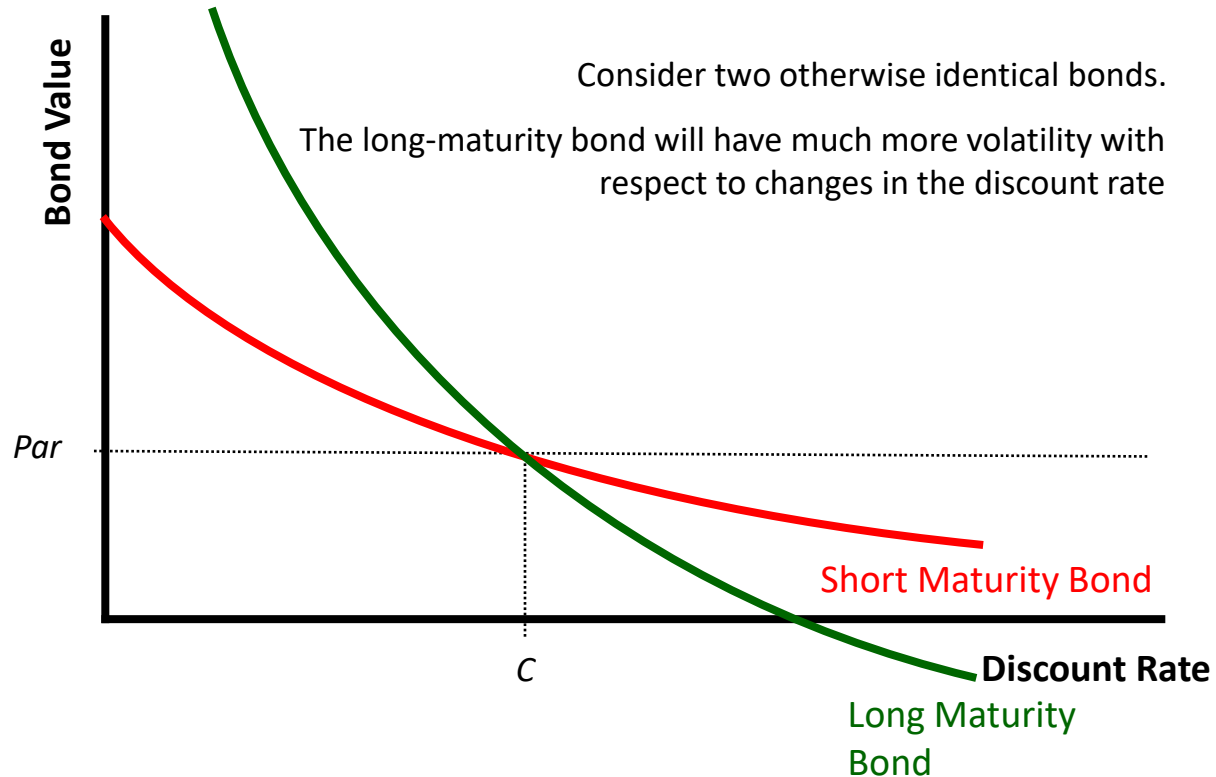
Volatility of a bond is the sensitivity of the bond expressed per unit of bond value. It coincides with the Modified Duration of the bond

PRICE VOLATILITY VS MATURITY/COUPON

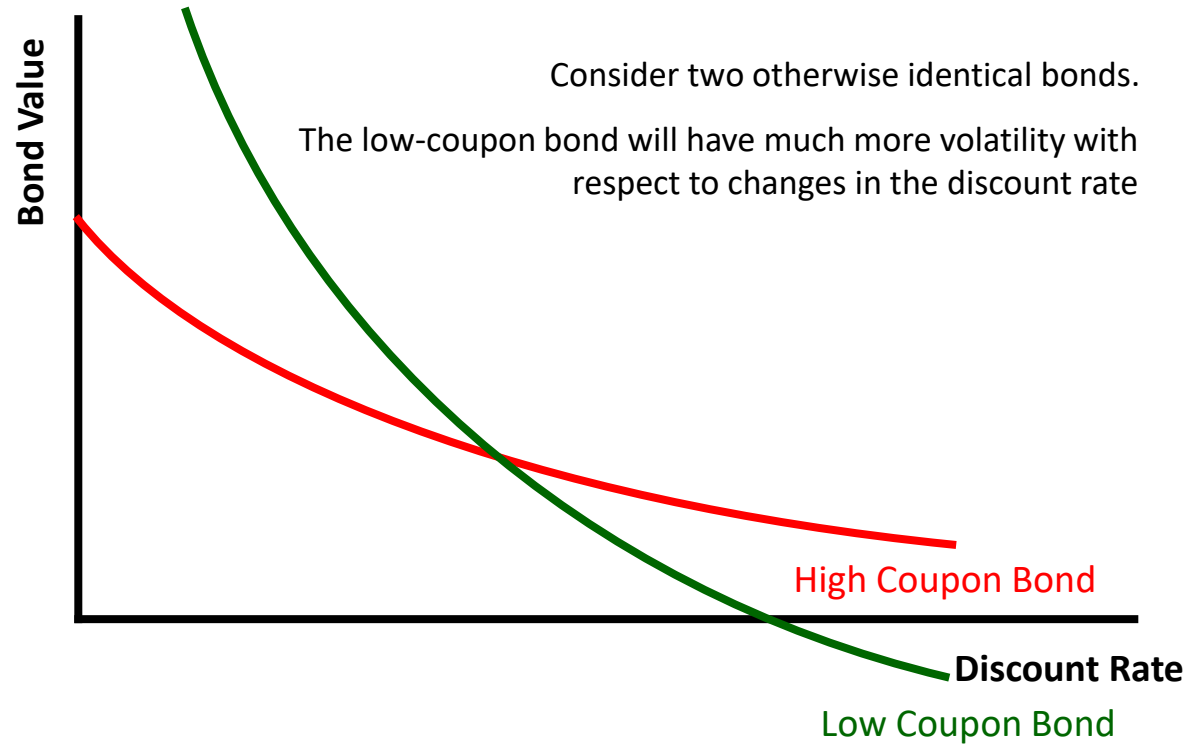
- As the **maturity of the bond increases**, the price volatility of premium bonds **increases in magnitude**.
- As the **coupon size of the bond increases**, the price volatility **decreases in magnitude**.



MATURITY AND BOND PRICE VOLATILITY



COUPON RATE AND BOND PRICE VOLATILITY



DISCOUNT BONDS: MATURITY & VOLATILITY

COUPON RATE: 12%, FACE VALUE 100



YTM	MATURITY	PRICE	PRICE SENSITIVITY	PRICE VOLATILITY
40	01	88.57		
41	01	87.94	63	0.71
40	02	71.84		
41	02	70.88	96	1.33
40	10	32.83		
41	10	31.93	90	2.75
40	100	30.00		
41	100	29.27	73	2.43

MATURITY AND BOND PRICE VOLATILITY: EXAMPLE

- **Two bonds X and Y are both 12% annual coupon bonds of the face value of 1,000. They are redeemed at par after two years and ten years respectively. Calculate the percentage change in price of each bond when the market interest rates change from 5% to 6% and from 25% to 26%.**



For Bond X

$$P = P(12\%, 5\%, 2) = 120PVIFA(5\%, 2)$$

$$+PVIF(5\%, 2) = 1130$$

$$P^* = P(12\%, 6\%, 2) = 120PVIFA(6\%, 2)$$

$$+PVIF(6\%, 2) = 1110$$

$$\text{Price Volatility} = -\frac{(P^* - P)}{P} = 1.78\%$$

Similarly, Price Volatility of Bond Y = 6.42%



RESULTS

- $P_X(12\%, 5\%, 2) = 1130$
- $P_X(12\%, 6\%, 2) = 1110$
- PRICE VOLATILITY OF X = 1.78%

- $P_Y(12\%, 5\%, 10) = 1541$
- $P_Y(12\%, 6\%, 10) = 1441$
- PRICE VOLATILITY OF Y = 6.42%



BOND PRICE VOLATILITY AND COUPON SIZE: EXAMPLE

- **Two bonds X and Y are 10% and 80% annual coupon bonds of the face value of 1,000. They are redeemed at par after thirty years. Calculate the percentage change in price of each bond when the market interest rates change from 5% to 6%.**



For Bond X

$$P = P(10\%, 5\%, 30) = 100PVIFA(5\%, 30)$$

$$+PVIF(5\%, 30) = 1769$$

$$P^* = P(10\%, 6\%, 30) = 100PVIFA(6\%, 30)$$

$$+PVIF(6\%, 30) = 1551$$

$$\text{Price Volatility} = -\frac{(P^* - P)}{P} = 12.32\%$$

Similarly, Price Volatility of Bond Y = 10.72%



RESULTS

- $P_X(10\%, 5\%, 30) = 1769$
- $P_X(10\%, 6\%, 30) = 1551$
- PRICE VOLATILITY OF X = 12.32%

- $P_Y(80\%, 5\%, 30) = 12523$
- $P_Y(80\%, 6\%, 30) = 11125$
- PRICE VOLATILITY OF Y = 10.72%



SHIFTS IN YIELD CURVE & EFFECTIVE & KEY RATE DURATION



RECALL: MACAULAY, MODIFIED DURATION & CONVEXITY

- **Modified duration measures a bond's price sensitivity to interest rate changes, assuming that the bond's cash flows do not change as interest rates change.**
- **The standard measure of convexity can be used to improve price changes estimated from modified duration.**

- **Duration is the linear approximation that considers only the first order term in the Taylor series.**
- **By considering convexity, we are accounting for the curvature of the yield-price curve through the second order term in the Taylor series.**

- $\frac{dP}{P} = -D_{Mac} \frac{dy}{1+y} + C \left(\frac{dy}{1+y} \right)^2$
- $= -D_{Mod} dy + C \left(\frac{dy}{1+y} \right)^2$

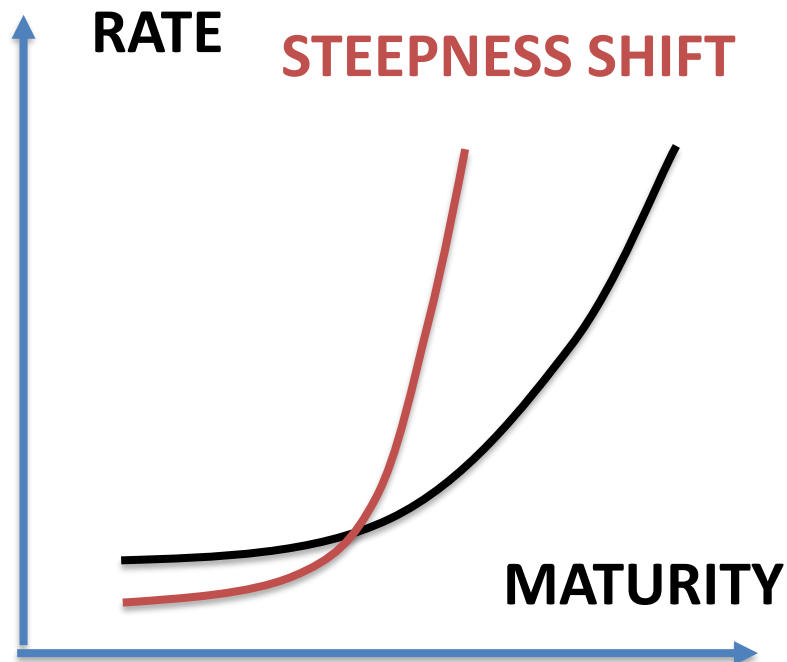
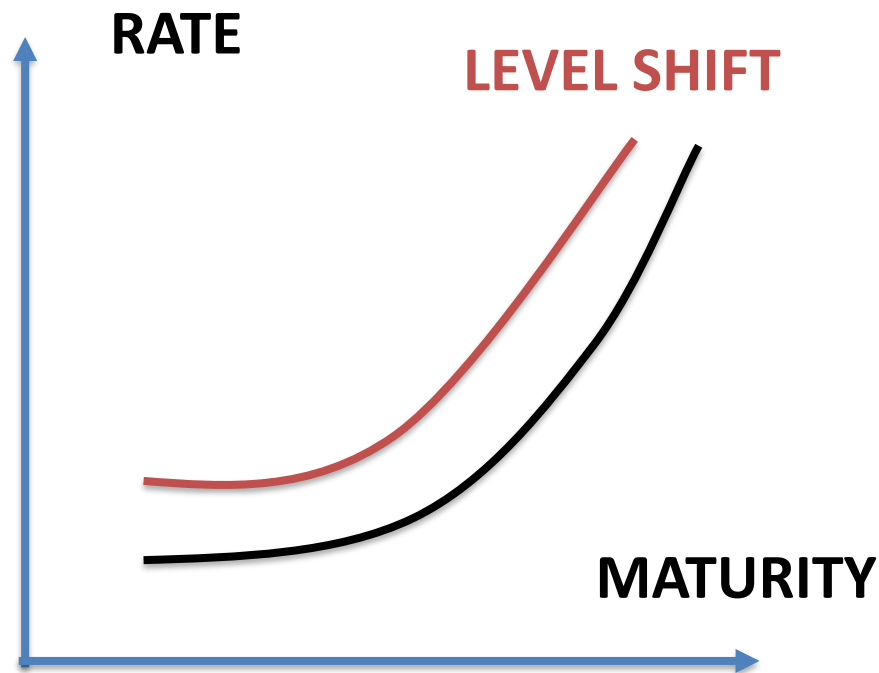
YIELD CURVE RISK & DURATION

- Yield curve risk refers to risk arising from change in value of a bond portfolio due to unanticipated changes in the yield curve.

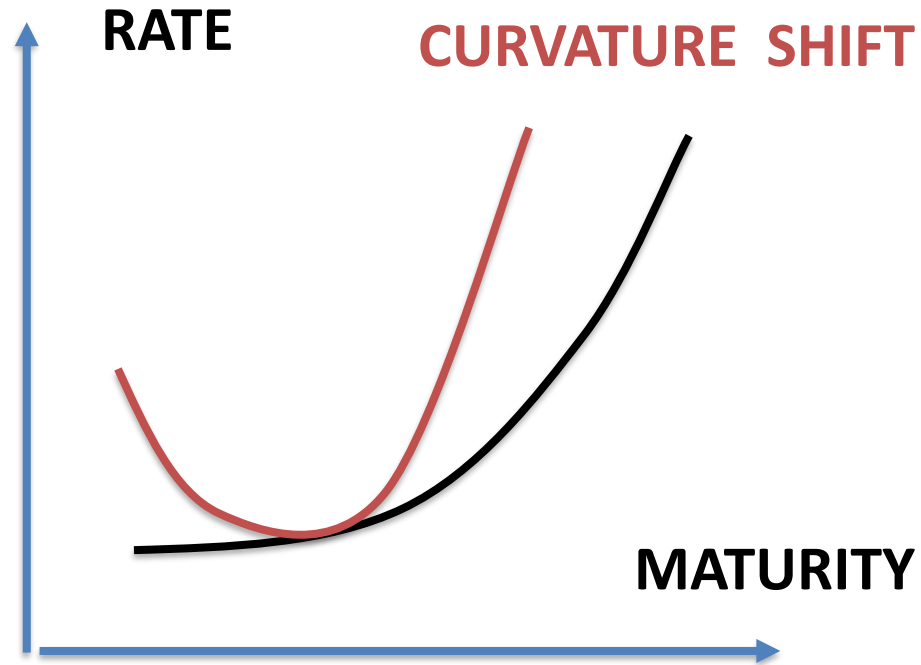
DECOMPOSITION OF YIELD CURVE SHIFTS

- Yield curve shifts can be decomposed into the following:
- Level (Δx_L) – A parallel increase or decrease of interest rates.
- Steepness (Δx_S) – Long-term interest rates increase while short-term rates decrease.
- Curvature (Δx_C) – Increasing curvature means short- and long-term interest rates increase while intermediate rates do not change.
- It has been found that all yield curve movements can be described using a combination of one or more of these movements.

SHIFTING OF YIELD CURVE



CURVATURE OF YIELD CURVE



- We can then model the change in the value of our portfolio as follows:
- $$\frac{\Delta P}{P} = -D_L \Delta x_L - D_S \Delta x_S - D_C \Delta x_C \quad (1)$$
- where D_L , D_S and D_C are respectively the portfolio's sensitivities to changes in the yield curve's level, steepness, and curvature.

SHAPING RISK

- Shaping risk refers to changes in portfolio value due to changes in the *shape* of the benchmark yield curve.
- It covers the non-parallel shifts in the yield curve.

YIELD CURVE SENSITIVITY

- **Yield curve sensitivity can be generally measured by**
- **effective duration, or**
- **key rate duration, or**
- **three-factor model that decomposes changes in the yield curve into changes in level, steepness, and curvature.**

